A P-STABLE LINEAR MULTISTEP METHOD FOR SOLVING STIFF DELAY DIFFERENTIAL EQUATIONS

Akinfenwa, O.A.[1], Abdulganiy, R.I.[2], Okunuga, S.A.[3] and Obinna U.K.[4]
[1,2,3,4] Department of Mathematics, University of Lagos, Lagos, Nigeria.

In this paper, a P-stable linear multistep method is derived for the numerical solution of delay differential equations (DDEs). The method has order k+1 and implemented using a constant step size in a block by block fashion. The P-stability region of the method is discussed. The numerical results revealed that the method is efficient and reliable for the solutions of first order delay differential equations. Tables 1-4 presented for some standard delay problems show the accuracy of the method when compared to those in the literature.

Keywords: Delay differential equation; Multistep method; P-stability region.

1. Introduction

Delay Differential Equations (DDEs) are a large and important class of dynamical systems. They often arise in either natural or technological control problems. Some of the known application areas of DDEs include the fields of Engineering, Biology and Economics, such as mixing of liquid, population growth, prey-predator population model and electrodynamics. The solution of DDEs requires the knowledge of not only the current state, but also of the state at a certain previous time. In this paper, we consider the numerical solution of a first order DDEs of the form:

$$y'(t) = f(t, y(t), y(t-\tau_1), y(t-\tau_2), \ldots, y(t-\tau_n)), \qquad t \geq t_0$$
$$y(t) = \varphi(t), \; t \leq t_0 \qquad (1)$$

where $\varphi(t)$ is the initial function which gives the behavior of the system prior to time t_0 (assuming that we start at $t_0 = 0$). and τ (t, y(t)), is the delay function and y(t - τ (t, y(t))) is the solution of the delay term. The delay function could either be constant, time dependent, $\tau_i = \tau_i(t)$ or state dependent $\tau_i = \tau_i$(t, y(t)), where i = 1; 2,…n (Bellen and Zennaro 2003).

In the literature, the numerical solution of DDE are of wide interest and many approaches which include the one step method such as Euler method, Runge-kutta methods as well as multistep methods. Among these approaches the Runge-kutta methods are the most popular numerical method for solving DDEs. (Oberle and pesh 1981), solved the DDEs based on the well-known Runge Kutta Fehlberg method where the retarded argument is approximated by an appropriate multipoint Hermite interpolation, (Ismail et. al. 2002), solved DDEs using embedded Singly Diagonally Implicit Runge-Kutta (SDIRK) method and Hermite interpolation was applied to evaluate the delay terms. (Bocharov et al. 1996), implemented the linear multistep methods for the numerical solution of initial value problem for stiff delay differential equations with several constant delays using the Nordsieck's interpolation technique for approximating the delayed variable. Recently, block method has been proposed for solving DDEs. (Majid et. al. 2013), solved delay differential equation by a five-point one-step block method based on the Newton backward divided difference formulae using Nevilles interpolation for solving delay term. (Heng et. al.2013) applied the 2-point Block Backward Differentiation Formula (BBDF) which achieved better results in terms of the accuracy as well as execution time when compared to 1-point BDF. However, all these methods were implemented either in a predictor corrector mode or by requiring superficial points or back values from other methods. In this paper the delay function defined over the interval [-t, 0] is mapped into the present function on the interval [0, T], then the equation (1) is solved by applying the P-stable method as a self-starting method to simultaneously provide the solution of first order DDEs.

2. Formulation of the Method

The block algorithm proposed in this paper is based on interpolation and collocation. We proceed by seeking an approximate of the exact solution $y(t)$ by assuming a continuous solution $Y(t)$ of the form

$$Y(t) = \sum_{j=0}^{k+1} \mu_j \varphi_j(t) \tag{7}$$

Such that $t \epsilon [t, T], m_j$ are unknown coefficients and $\varphi_j(t)$ are polynomial basis functions of degree k+1. The integer $k = 2$ denotes the step number of the method. We thus, construct a 2-step Block Methods with $\varphi_j(t) = t_{n+I}^j$ by imposing the following conditions

$$\sum_{j=0}^{3} \mu_j t_{n+i}^j = y_{n+i} \qquad i = 0,1 \tag{8}$$

$$\sum_{j=0}^{3} \mu_j j t_{n+i}^{j-1} = f_{n+i} \qquad i = 2,3 \tag{9}$$

where y_{n+j} is the approximation for the exact solution $y(t_{n+j})$, $f_{n+j} = f(t_{n+j}, y_{n+j})$ n is the grid index and $t_{n+j} = t_n + jh$. It should be noted that equation (8) and (9) leads to a system of k + 2 equations of the form $AU = C$ where

$$A = \begin{pmatrix} 1 & t_n & t_n^2 & t_n^3 \\ 1 & t_{n+1} & t_{n+1}^2 & t_{n+1}^3 \\ 0 & 1 & 2t_{n+2} & 3t_{n+2}^2 \\ 0 & 1 & 2t_{n+3} & 3t_{n+3}^2 \end{pmatrix}$$

$$U = (u_0, u_1, u_2, u_3)^T, \quad C = (y_n, y_{n+1}, f_{n+2}, f_{n+3})$$

which is then be solved to obtain the coefficientμ_j. After some algebraic computations our *k*-step continuous block p-stable method is then obtained by substituting these values of μ_j into equation (7). We then obtained the expression in the form

$$Y(t) = \sum_{j=0}^{1} \alpha_j(t) y_{n+j} + h\beta_k(t) f_{n+k} + h\beta_{k+1}(t) f_{n+k+1} \tag{10}$$

where $\alpha_j(t)$, $\beta_k(t)$ and $\beta_{k+1}(t)$ are continuous coefficients. The method (10) is then used to generate the standard two step method of order (k+1) at the desired point $t = t_{n+i}$, $i = 2,3$ The additional method is then obtained by evaluating the first derivative of (10) given by (11) at $t = t_{n+1}$ number of points.

$$Y'(t) = \frac{1}{h}\left(\sum_{j=0}^{1} \alpha'_j(t) y_{n+j} + h\beta'_k(t) f_{n+k}\right) + h\beta'_{k+1}(t) f_{n+k+1}) \tag{11}$$

This additional integrators (11) are combined with the main methods (10) and implemented as a self starting block method given as:

$$y_{n+2} = \frac{1}{23}[22hf_{n+2} - 4hf_{n+3} - 5y_n + 28y_{n+1}]$$

$$y_{n+3} = \frac{1}{23}[36hf_{n+2} + 6hf_{n+3} - 4y_n + 27y_{n+1}]$$

$$f_{n+1} = \frac{1}{23}[16hf_{n+2} - 5hf_{n+3} - 12y_n + 12y_{n+1}] \tag{12}$$

3. Analysis of the New Method

3.1 Local Truncation Error

Following (Lambert 1991) the local truncation error associated with each of the method in the p-stable multistep method can be defined to be the linear difference operator

$$L[\,y(t)\,;h\,] \;=\; \sum_{j=0}^{3}\alpha_j y_{n+j} - h\sum_{j=2}^{3}\beta_j f_{n+j} \qquad (13)$$

Assuming that y(t) is sufficiently differentiable, we can write the terms in (13) as a Taylor series expression of $y(t_{n+j})$ and $f(t_{n+j}) = y'(t_{n+j})$ as

$$y(t_{n+j}) = \sum_{j=0}^{\infty}\frac{(jh)}{m!}y^{(m)}(t_n) \text{ and } y'(t_{n+j}) = \sum_{j=0}^{\infty}\frac{(jh)^m}{m!}y^{(m+1)}(t_n) \qquad (14)$$

Substituting (14) into the equations in (13) we obtain the expression

$$L[\,y(t_n)\,;h\,] = C_0 y(t) + C_1 h y'(t) + C_2 h^2 y''(t) + \ldots + C_m h^m y^m(t) + \ldots$$

where the constant C_m , $m = 0, 1, 2, \ldots$ are given as follows:

$$C_0 = \sum_{J=0}^{k}\alpha_j$$

$$C_1 = \sum_{J=1}^{k} j\,\alpha_j - \beta_k - \beta_{k+1} + \gamma_l$$

$$C_2 = \frac{1}{2!}\sum_{J=1}^{k} j^2\,\alpha_j - 2\beta_k - 3\beta_{k+1} + \gamma_l$$

$$C_m = \frac{1}{m!}\left[\sum_{J=1}^{k} j^m\,\alpha_j - m(k)^{m-1}\beta_k - m(k+1)^{m-1}\beta_{k+1} + ml^{\,m-1}\,\gamma_l\right]$$

where $\gamma_l = 1$, $l = 1$

The new block method in (13) is said to have a maximal order of accuracy m if

$$L[\, y(t_n)\,;h\,] = C_{m+1}h^{m+1}y^{m+1}(t_n) + O(h^{m+2})$$

and

$$C_0 = C_1 = C_2 \ . \ . \ . C_m = 0 \ , \quad C_{m+1} \neq 0 \qquad (15)$$

Therefore, C_{m+1} is the error constant and $C_{m+1}h^{m+1}y^{m+1}(t_n)$ the principal local truncation error at the point t_n. From our analysis we have that the new method has its local truncation error as $\left(\frac{17}{138}, \frac{3}{46}, -\frac{9}{46}\right)^T$ with order $(3, 3, 3)^T$

In what follows, the new method can be generally rearranged and rewritten as a matrix finite difference equation of the form

$$A_1 Y_{N+1} = A_1 Y_N + hU_1 F_{N+1} \qquad (16)$$

where,

$$A_0 = \begin{bmatrix} 0 & 0 & \frac{-12}{23} \\ 0 & 0 & \frac{-5}{23} \\ 0 & 0 & \frac{-4}{23} \end{bmatrix}, \ Y_N \begin{bmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix}, \ A_1 = \begin{bmatrix} \frac{-12}{23} & 0 & 0 \\ \frac{-28}{23} & 1 & 0 \\ \frac{-27}{23} & 0 & 1 \end{bmatrix}, \ Y_{N+1} \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{bmatrix}$$

$$U_1 = \begin{bmatrix} -1 & \frac{16}{23} & \frac{-5}{23} \\ 0 & \frac{22}{23} & \frac{-4}{23} \\ 0 & \frac{36}{23} & \frac{6}{23} \end{bmatrix}, \ F_{N+1} = \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{bmatrix}$$

3.2 Zero Stability of the method

The zero stability of the method is concerned with the stability of the difference system in the limit as h tends to zero (Lambert 1991). Thus, as $h \to 0$ the difference system (16) tends to

$$A_1 Y_{N+1} = A_0 Y_N \quad ,$$

whose first characteristics polynomial $\rho(R)$ given by

$$\rho(R) = \det\left[RA^{(1)} - A^{(0)}\right] = \frac{12}{23} R^2 (1-R) \quad , \qquad (17)$$

The new block method (13) is zero stable for $\rho(R)=0$ and satisfies $|R_j| \leq 1$, $j = 1, \ldots k$, and for those roots with $|R_j| = 1$, the multiplicity does not exceed 1. hence the block method with continuous coefficients are zero stable.

3.3 Consistency and Convergence

We note that the new block method (16) is consistent as it has order $m > 1$. Since the block method (16) is zero stable and Convergence = zero stability + consistency, hence the method (16) converges.

3.4 Linear Stability of the Method

There are many concepts of linear stability of numerical methods when applied to DDEs, depending on the test equation as well as the delay term involved. We will discuss the stability analysis of block method for the numerical solution of delay differential equations (DDEs). using the following linear test equations of the form,

$$\begin{aligned} y'(x) &= \lambda y(x) + \mu y(x-\tau) \qquad x \geq x_0 \\ y(x) &= \varphi(x) \qquad -\tau \geq x \geq x_0 \end{aligned} \qquad (18)$$

where λ and μ are complex numbers.We consider a fixed step size $x_n = x_0 + nh$ $H_1 = h\lambda$, $H_2 = h\mu$ and $mh = \tau$, $m \in I^+$ Such test equation has been used by (Barwell 1975) (Al-Mutib 1984) and (Zennaro 1986). Introduced the concept of P stability.

Definition 3.1 (P-*stability*)

For a fixed size h, if λ and, μ are real in (13), the region R_P in (H_1,H_2)-plane is called P-stability if for any $(H_1,H_2) \in R$ the numerical solution satisfies (13) as $x \to \infty$.

Applying (16) to test equation (18) we obtain equation (19) given in the form

$$A_1 Y_{N+1} = A_0 Y_N + hU_1 (\lambda Y_{N+i} + \mu Y_{N+i-m}) \qquad (19)$$

Simplifying equation (19) becomes

$$(A_1 - H_1U_1)Y_{N+1} - A_0Y_N - H_2U_1Y_{N+1-m} = 0 \qquad (20)$$

Where $H_1 = h\lambda$ and $H_2 = h\mu$

Thus, the P-stability polynomial is obtained as ,
$\psi(t) = \det[\,(A_1 - H_1B_2)t^{3+m} - A_0t^{2+m} - H_2B_1t^{2+m}]$

The P-stability region when m=1 for the block method is shown in the figure 1 below

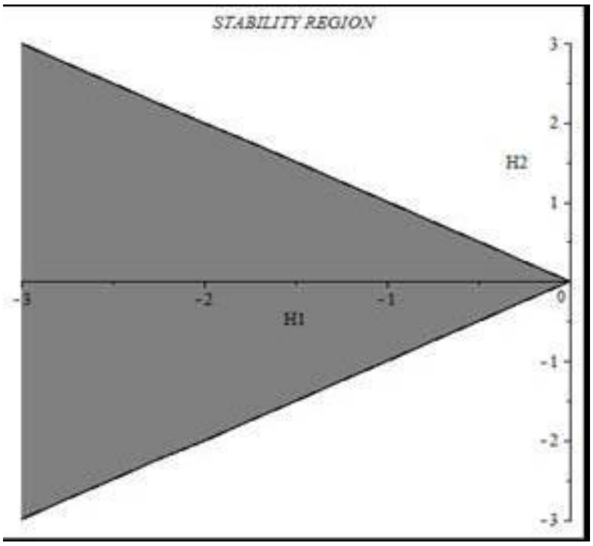

Fig. 1: P-stability region

4. Numerical Examples

In this section, we give numerical example to illustrate the accuracy of the block method. The computations are carried out using our written code in maple 17.

PROBLEM 1: $y'(t) = -1000y(t) + y(t - (\ln(1000-1))), \qquad 0 \le t \le 10$
$y(t) = e^{-t}, \; t < 0$

Exact solution is $y(t) = e^{-t}$,

This problem has also been solved by (S. C Heng *et al* 2013). Their results are here reproduced together with the results obtained using the two BBDF in

(Akinfenwa et al.2012). In Table 1, 2, 3 and for a constant step size h the maximum absolute error is compared with the results obtained with the new block method

Table 1: Comparison of Max. error $=|y-y_i|$ **for problem 1**

h	METHOD	MAX ERROR
10^{-2}	2BBDF	$3.80236e^{-004}$
10^{-2}	Two step BBDF	$6.13171e^{-006}$
10^{-2}	New method	$4.61670e^{-008}$

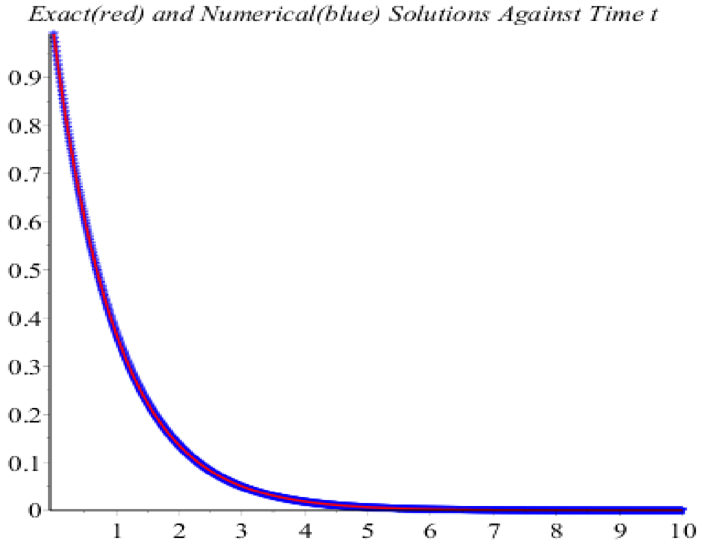

Fig 2: Graph of Exact against numerical using the new method

PROBLEM 2:

$$y'(t) = -1000y(t) + 997e^{-3}y(t-1) + 1000 - 997e^{-3}), \qquad 0 \le t \le 10$$
$$y(t) = 1 + \exp(-3t)\,, \quad t \le 0$$

Exact solution is $y(t) = 1 + \exp(-3t)$.

Table 2: Comparison of Max. Error $=|y-y_i|$ for problem 2

h	METHOD	MAX ERROR
10^{-2}	2BBDF[11]	$3.40594e^{-003}$
10^{-2}	Two step BBDF[1]	$5.52039e^{-005}$
10^{-2}	New method	$1.25653e^{-06}$

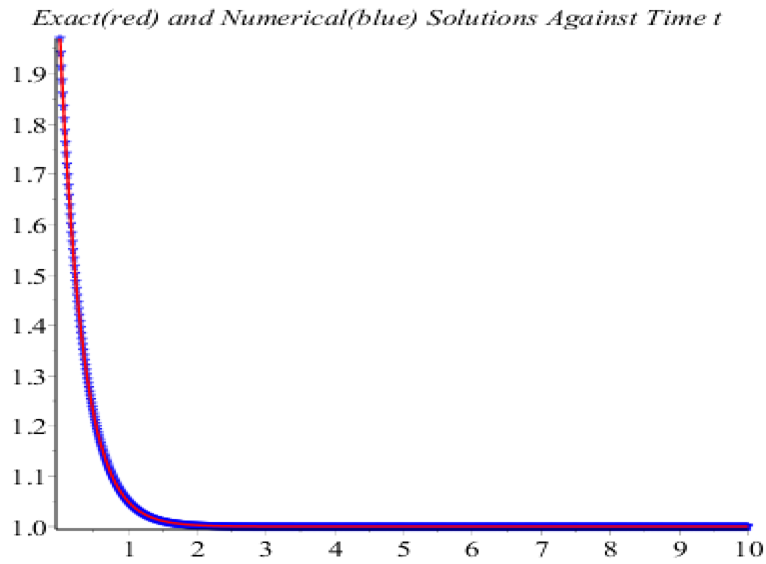

Fig. 3: Graph of Exact against numerical using the new method

PROBLEM 3: Consider the second order delay differential equation which has been solved by Suha et.al [10] using modified fourth order Runge-Kutta method (RK4SY)..

$$z''(t)+\frac{1}{2}y(t)-\frac{1}{2}y(t-3\pi)=0 \qquad t\geq t_0$$

With initial function $y(t)=Cos\ t \qquad -3\pi\leq t\leq 0$

The above delay differential equation can be replaced with a system of two first order delay differential equation given by

$$y_1'(t) = y_2(t) \qquad\qquad y_1(t) = Cos\ t$$

$$y_2'(t) = -\frac{1}{2}y_1(t) + \frac{1}{2}y_1(t-3\pi) = 0 \qquad y_2(t) = -Sin\ t$$

$$-3\pi \le t \le 0$$

Table 3: Comparison of Max. Error $=| y - y_i |$ **for problem 3**

h	METHOD	MAX ERROR
10^{-1}	RK4SY[10]	$2.00000e^{-004}$
10^{-1}	New method	$7.282153e^{-005}$

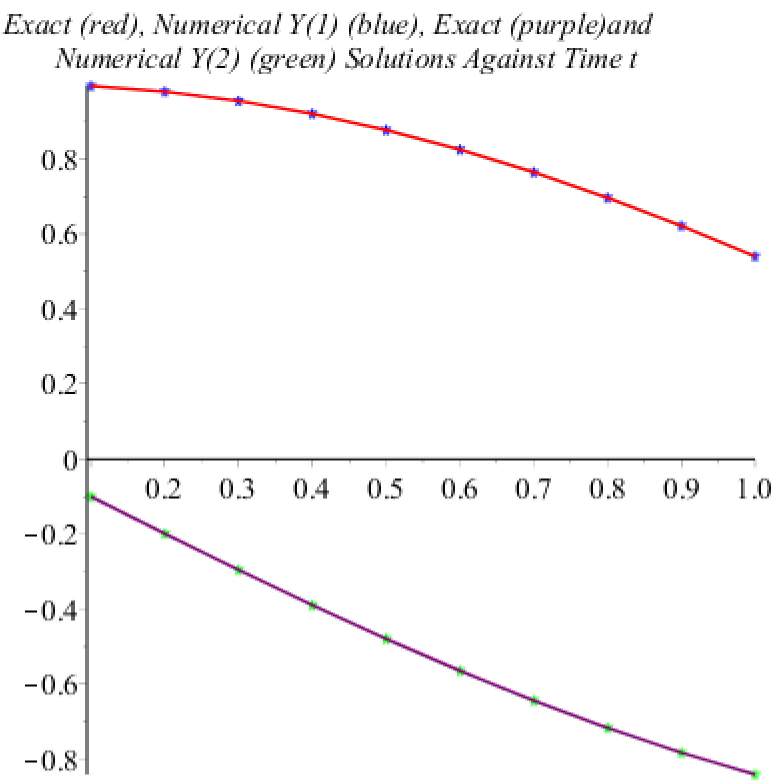

Fig. 4: Graph of Exact against numerical using the new method

Table 4: Comparison of Methods

t	$Exact_1$	RK4Sy 1[10]Y_1	BBDF [1]Y_1	New methodY_1	$Exact_2$	RK4Sy 1[10]Y_2	BBDF[1] Y_2	New method Y_2
0.1	0.9950	0.9951	0.9950	0.9950	-0.0998	-0.0998	-0.1002	-0.0998
0.2	0.9801	0.9802	0.9799	0.9801	-0.1987	-0.1986	-0.1987	-0.1987
0.3	0.9553	0.9555	0.9550	0.9553	-0.2955	-0.2954	-0.2962	-0.2955
0.4	0.9211	0.9212	0.9207	0.9211	-0.3894	-0.3893	-0.3899	-0.3894
0.5	0.8776	0.8778	0.8769	0.8776	-0.4794	-0.4792	-0.4802	-0.4794
0.6	0.8253	0.8255	0.8247	0.8253	-0.5646	-0.5644	-0.5643	-0.5646
0.7	0.7648	0.7650	0.7638	0.7648	-0.6442	-0.6440	-0.6451	-0.6442
0.8	0.6967	0.6969	0.6956	0.6967	-0.7174	-0.7171	-0.7181	-0.7174
0.9	0.6216	0.6218	0.6201	0.6216	-0.7833	-0.7831	-0.7842	-0.7833
1.0	0.5403	0.5405	0.5388	0.5403	0.8415	0.8413	0.8421	0.8415

5. Conclusion

In this paper a P-stable multistep method for the solution of first order Delay differential equations has been presented. The block method is used to simultaneously to solve (1) by first mapping the initial function into the present function. The efficiency of the method has been demonstrated on some standard delay problems and compared with existing schemes in the literature. Details of the numerical results are displayed in Tables 1-4 and their graphical solution presented. Future work will focus on the direct solution of higher order delay differential equations.

References

Akinfenwa, O.A. Jator, S.N. and Yao, N.M. (2012). On the stability of continuous, Block method Backward Differentiation Formula For solving stiff ordinary Differential Equation, J. of Mod. Meth. in Numer. Math vol. 3 (2) pp 50-55.

Al-Mutib, A.N. (1984). Stability properties of numerical methods for solving delay differential equations, J. Comput. Appl. Math. 10 (1) pp. 71-7.

Barwell,V.K., (1975) Special stability problems for functional equations, BIT 15 pp.130-135.

Bellen, A. and Zennaro, M. (2003). Numerical Methods for Delay Differential Equations New York: Oxford University Press.

Bocharov, G. A., Marchuk, G. I., and Romanyukha, A. A. ,(1996). Numerical solution by LMMs of Stiff Delay Differential systems modelling an Immnune Response. Numer. Math.,73 pp 131-148.

F. Ismail, R.A Al-Khasawneh, A.S Lwin, and M.B Suleiman. (2002). Numerical treatment of delay differential equations by Runge-Kutta method using Hermite interpolation, Mathematika 18 pp. 79-90. .

Heng, S. C., Ibrahim, Z. B., Suleiman, M. and Ismail, F. (2013). Solving delay differential equations by using implicit 2-point block backward differentiation formula Pertanika J. Sci. Technol. 21 (1), 37 - 44.

Lambert J.D. (1991), *Numerical Methods for Ordinary Differential Systems*, John Wiley, New York.

Majid, Z. A., Radzi, H. M. and Ismail, F. (2013) Solving delay differential equations by the five-point one-step block method, International Journal of Computer Mathematics using Neville's interpolation DOI:10.1080/00207160.2012.754015.

Oberle, H.J. and Pesh, H.J. (1981). Numerical treatment of delay differential equations by Hermite interpolation. Numer. Math37 pp.235-255.

Sahu, N. A. and Raghad, K.S. 2006, Numerical solution of Nth order linear delay differential equations using Runge-Kutta method Um- Salama Science Journal Vol3(1) pp 140-146

Zennaro, M. (1986). P-stability properties of Runge-Kutta methods for delay differential equations, Numer. Math.49 pp.305-318. .