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# MODELLING CRUDE OIL PRICE TIME-VARYING VOLATILITY USING JUMP-DIFFUSION MODEL

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## ABSTRACT
*The objective of this paper is to capture the time-varying volatility in crude oil prices. The time-varying volatility dynamics are characterized by high volatility, high intensity jumps, and strong upward drift, indicating that oil markets were constantly out-of-equilibrium. The method of maximum likelihood and cumulants are utilized. The Jump-Diffusion model, generalized autoregressive conditional heteroskedasticity (GARCH) model and autoregressive model of order two (AR(2)) are used to empirically model the crude oil price (January, 1986-July, 2015). The results show that the three models performed well in estimating the crude oil price. However, Jump-Diffusion model out-performed the other two models as it captures the drift as well as the jumps in the crude oil price within the sampled period. The results establish that commodity price risk plays a dominant role in the energy industries, and the use of derivatives has become a common means of helping energy firms, investors and customers to manage risks that arise from the high volatility.*

***Key words:*** Volatility, Cumulants, Mean reversion, Diffusion process, Stochastic differential equation

## INTRODUCTION
Oil embodies a vital role in the national and international economies as the backbone and the source of raw inputs for numerous industries. It is an important source of energy and represents indispensable raw material as a major component in many manufacturing processes and transportation fuel (Gabralla and Abraham, 2013). Strong growth in the demand for oil worldwide, particular in China and other developing countries is generally accepted as a driving force behind the sharp price volatility seen over the past seven years. Oil prices tend to exhibit strong seasonal patterns in response to cyclical fluctuations in supply and demand mostly due to weather and climate changes (Krichene (2006), Agwuegbo *et al*., (2017)). Despite the sharp rises during short periods of such specific events, oil

prices usually revert to a normal level. This means, oil prices will fluctuate around and drift over time to values determined by the cost of production and the level of demand.

The dynamics will essentially depend on the economic environment, in our case on conditions of the global oil market, like supply and demand. Especially, in the latest months of 2006 - 2009, there was a significant pressure primarily due the growing demand for crude oil from Asia driving the oil price to heights not reached ever before. However, within the natural zone, prices may strongly oscillate by crossing different domains of attractions several times mainly due to random perturbations (e.g. war or political or economic instabilities, or environmental conditions). While developments in crude oil prices were being followed closely by economic agents, including traders, investors, speculators, and policy makers, not much was known about the stochastic processes driving these prices. Crude oil prices, in spite of their importance, have recently attracting extensive academic and research works. The problems and techniques for construction of dynamical models from noisy chaotic time series is given to supplement existing surveys due to the use of a special systematization of the variety of problem settings and methods (Kantz and Schreiber (1997). Mathematical modelling is a rational approach to a better understanding of real processes, and in general an approximation to reality, which can help to analyze consequences of our assumptions on reality.

## RELATED WORK

Several academic and commercial research teams are very active in the crude oil price dynamics such as high volatility, high intensity jumps, and strong upward drift, indicating that oil markets were constantly out-of-equilibrium. In global markets, oil is the most active and heavily traded commodity. Oil price suffer from high volatility and fluctuations. Andersen et al. (2007), Boudth *et al*. (2011), and Lee and Mykland (2008) assume that the log-prices are observed without measurement error. It is however more realistic to consider that the logarithm of the recorded asset price is actually the sum of the logarithm of the so-called “efficient” price and a noise component that is induced by microstructure frictions. Ait-Sahalia *et al*. (2011) divide the sources of microstructure noise into three groups: (i) frictions inherent in the trading process such as discreteness of price changes and rounding; (ii) informational effects such as the gradual response of prices to a block trade or inventory control effects and (iii) measurement or data recording errors such as prices entered as zero or misplaced decimal points. Wang *et al*. (2010) attributed the volatility of oil prices to three main factors: increase in demand and supply shortages possibly caused by economic growth or the behaviours of oil producing countries; exogenous events such as wars, natural disasters, etc. and endogenous factors such as speculations in the markets. Ghosh (2001) investigates impact of crude oil shocks on exchange rate link for India and observed that an increase in the oil price return leads to the low price of Indian currency compare to the United States (US) dollar and oil price shocks have the continuous effect of exchange rate volatility.

Generally, high crude oil prices directly affect the cost of gasoline, home heating oil, manufacturing and electric power generation. This makes the world to always in want of crude oil, so the needs for oil continue to the rise and production continues to fall in 1999, the price of crude oil ranged about \$16 a barrel. Between 2008 and 2009, the crude oil price passed the \$100 a barrel and fluctuated between \$147.96 and \$69, (Gabralla and Abraham, 2013). These unprecedented shock and wide fluctuations have significant impact plus negative effects on petroleum exporting countries and consuming oil countries, (Gibson and Schwartz, 1990). Empirical studies strongly suggest that volatility is not constant, but has a random component. ARCH/GARCH models, whose continuous-time diffusion limits are stochastic volatility models, provide much better descriptions of the data. See Duffie and Singleton (2000).

A generalized autoregressive conditional heteroskedasticity (GARCH) (1, 1) model is defined as follows: The mean equation:

$$x_t = \Delta \log S_t = c + \varepsilon_t, \ \ \varepsilon_t \sim \mathrm{N}(0, \ \sigma_t^2)\,. \tag{1}$$

The conditional variance equation:

$$\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \tag{2}$$

where $\sigma_t^2 = \mathrm{E}(\varepsilon_t^2)$.The parameters: $\alpha_0 > 0,\ \alpha_1 \geq 0$ and $\beta_1 \geq 0$ are strictly positive to ensure the positivity of the conditional variance, $\sigma_t^2$. The GARCH (1, 1) model captures the empirical characteristics such as the presence of skewness and kurtosis in oil price returns data where only the magnitude is considered and not the direction of the crude oil price. Since there is the presence of jumps in asset prices and for more accurate option pricing, we proffer a jump-diffusion process. Moreover, it is well-known that short-term options have market implied volatilities that exhibit a significant skew across strikes (Merton, 1976). The pure diffusion based models have difficulties explaining the similar effect in short-dated option prices, thus there is need in adding a jump component in modelling asset price dynamics (Bakshi *et al.*, 1977). Most importantly, the diffusion based stochastic volatility models cannot explain skewness in implied volatilities, except under implausible values for the model's parameters. Models with jumps generically lead to significant skews for short-term maturities. More generally, adding jumps to returns in diffusion based stochastic volatility model, the so-obtained model can generate sufficient variability and asymmetry in the short-term returns to match implied volatility skews for short-term maturities (Bates, 1966).

The key idea behind oil pricing models is to find partial differential equations that will solve for the price of oil futures contract. We restrict ourselves in this paper to a price dynamics modelled by just one stochastic differential equation with a stochastic volatility, controlled by a stochastic process. A first intuitive model for the dynamics of crude oil prices is a system of ordinary differential equations (deterministic approach):

$$d\mathrm{X} = \mu(t, \mathrm{X}, \mathrm{Y})dt \tag{3}$$

*Y*(t) describes external, e.g. economic or political effects. The oil price X(t) is a solution of the ordinary differential equations (3). The function $\mu$ could be a polynomial or a rational function in X with coefficients depending on *t* and X(t).

Consider here a perturbation caused by adding to equation (3) a stochastic process as

$$d\mathrm{X} = \mu(t, \mathrm{X}, \mathrm{Y})dt + \sigma dW(t, \mathrm{X}, \mathrm{Y}) \qquad t \geq 0 \tag{4}$$

where $\mu$ is the mean-reverting, $\sigma$ is the degree of volatility around the mean which is caused by stochastic shocks (random fluctuations of the underlying process), *W* denotes a Wiener process (Brownian motion). The functions, $\mu(\cdot)$ and $\sigma^2(\cdot)$ are called the drift and the diffusion functions of the price process. First of all, the functions $\mu$ and $\sigma$ are unknown and need to be determined by modelling and by means of data. If equation (4) has a zero mean, the crude oil spot price dynamics is assumed to follow basically a Geometric Brownian motion (continuous-time) which is seen as a solution to stochastic differential equations (SDEs):

$$d\mathrm{X} = \mu \mathrm{X} dt + \sigma \mathrm{X} dW \qquad t \geq 0 \tag{5}$$

where both $\mu$ and $\sigma$ are constant. For any arbitrary initial value $\mathrm{X}_0 = x_0$, the analytic solution is given as:

$$\mathrm{X}(t) = \exp\left(\frac{\mu - \sigma^2}{2}\right)t + \sigma W(t)x_0 \tag{6}$$

In many cases, this approach yields to explicit solutions which we usually do not obtain for a more complex dynamics as observed in energy and commodities prices that experience significant deviations from log-normality. To understand the price dynamics, it is worthwhile to analyze the ordinary differential equation that is focusing on the drift term. If X is higher (lower) than the long run equilibrium $\mathrm{X}_1$, the sign is negative (positive). Thus, the price is always reverting to its attracting level. Market data indicate that volatility exhibits mean-reverting behavior and this led Stein and Stein (1991) to introduce the mean-reverting Ornstein-Uhlenbeck process. Mean reversion in commodity prices has an important effect beyond its immediate impact on returns since it suggests that the implied volatility of commodity options will exhibit a downward-slopping term structure. As a consequence, the mean reversion process is probably the most common price model used by oil market practitioners and is often convenient to be modeled as an Ornstein-Uhlenbeck process (Agwuegbo *et al*., 2017). When crude oil prices are relatively high, existing producers will increase their production rate and new producers will enter the market, whereas consumers will ||at first replenish their stocks thereby, creating a downward pressure on prices. As long as the price is higher than the equilibrium price, this downward pressure is expected to last. Again, when prices are relatively low supply will decrease, since, for instance some of the high cost producers will exit the market and demand increases enforcing an upward pressure on prices. The Ornstein-Uhlenbeck process is somehow the most canonical choice for a random process and has been applied in finance and economics as the unique family of the continuous Markov processes with a stationary Gaussian distribution. In case of a constant volatility $\sigma(t,\mathrm{X},\mathrm{Y})=\sigma_0$ we obtain the Ornstein-Uhlenbeck model. Equation (3) can be interpreted as an Ito stochastic differential equation

$$\mathrm{X}(t)=e^{-K_0 t}x_0+\sigma_0 e^{-Kt}\int_0^t \exp(K_0 s)dW(s) \tag{7}$$

Equation (7) is usually referred to as Langevin stochastic difference equation and a linear Ito stochastic differential equation; it is also a Gaussian process.

Letting $\mathrm{X}_t=\log(S_t)$ and using Ito's lemma, the log price return process becomes

$$d\mathrm{X}_t=[\alpha-\tfrac{1}{2}\sigma^2]dt+\sigma d\mathrm{B}_t+J_t d\mathrm{N}_t=\mu dt+\sigma d\mathrm{B}_t+J_t d\mathrm{N}_t \tag{8}$$

where $\mu=\left(\alpha-\frac{1}{2}\sigma^2\right)$. The parameter vector associated with the price process is therefore $\theta=(\mu,\ \sigma^2,\ \lambda,\ \beta,\ \delta^2)$. Discretized over $(t,\ t+\Delta)$, the model takes the form:

$$\Delta\mathrm{X}_t=\mu\Delta+\sigma\Delta\mathrm{B}_t+\sum_{i=0}^{\Delta\mathrm{N}_t}J_i \tag{9}$$

where $\Delta B_t = B_{t+\Delta} - B_t \sim N(0,\ \Delta)$ and $\Delta N_t = N_{t+\Delta} - N_t$ is the actual number of jumps occurring during the time interval $(t,\ t+\Delta)$ and $J_t$ are independently and identically distributed as $J_i \sim N(\beta,\ \delta^2)$. The log-return $x_t = \Delta X_t$ includes therefore the sum of two independent components: a diffusion component with drift and a jump component. Its probability density is a convolution of two independent random variables ($\sigma^2$ and $\delta^2$) and can be expressed as

$$f(x) = \sum_{i=0}^{\infty} \frac{(\lambda\Delta)^n e^{-\lambda\Delta}}{n!}\left[\frac{1}{\sqrt{2\pi(\sigma^2\Delta + n\delta^2)}}\exp\left(-\frac{(x-\mu\Delta-n\beta)^2}{2(\sigma^2\Delta+n\delta^2)}\right)\right] \tag{10}$$

with $n = 0,\ 1,\ 2,\ \cdots$ Putting $\Delta = 1$, i.e., the time interval is ($t$, $t$ +1), the density function becomes

$$f(x) = \sum_{i=0}^{\infty} \frac{(\lambda)^n e^{-\lambda}}{n!}\left[\frac{1}{\sqrt{2\pi(\sigma^2 + n\delta^2)}}\exp\left(-\frac{(x-\mu-n\beta)^2}{2(\sigma^2+n\delta^2)}\right)\right] \tag{11}$$

Most modifications are concerning the modelling of the volatility function. A major approach in energy markets is the Pilipovic model. Pilipovic (1997) proposed a linear volatility function of the form: $\sigma(t, X, Y) = \sigma_0 X$. In addition, he allows the long-term equilibrium price $X_1$ to be driven by a secondary stochastic differential equation leading to the following two-factor model:

$$\begin{aligned} dX_1 &= K_1(X_2 - X_1)dt + \sigma_1 X_1 dW_1 \\ dX_2 &= K_2 X_2 dt + \sigma_2 X_2 dW_2 \end{aligned} \tag{12}$$

where $K_i, \sigma_i \quad i = 0,\ 1$ are positive constants, and $dZ$ determines the stochastic perturbation in the equilibrium price. Both, a pure Geometric Brownian motion as well as a simple mean reversion model are not in part able to capture fundamental phenomena of energy and commodity markets, so an additional random jump component is added to equation (4) to obtain Jump-Diffusion (J-D) models:

$$dX = \mu(t, X, Y)dt + \sigma(t, X, Y)dW + JXdQ(\lambda) \tag{13}$$

where $J$ is the jump component and $dQ$ is the Poisson process with parameter $\lambda$. Its value depends on the probability of occurrence of a jump, the expected size, and their expected standard deviation. Again it is possible to work with different assumptions on drift and volatility functions. In this context of oil markets it seems to be reasonable to consider a mean reversion process. Accordingly, the oil price evolves with mean reverting drift and two random terms: a diffusion and a

Poisson process embodying a random jump, (Geman, 2005). The arrival of jumps is governed by a Poisson process $dQ$ with arrival frequency parameter, $\lambda$. The proportional jump size $J$ may be a constant or drawn from a probability distribution.

Accordingly, the continuous-time stochastic process driving crude oil prices can be stated as a J-D process given by a stochastic differential equation

$$\frac{dS_t}{S_t} = \alpha dt + \sigma d\mathrm{B}_t + \left(\exp(J_t) - 1\right) d\mathrm{N}_t \tag{14}$$

where $S_t$ denotes the crude oil price, $\alpha$ is the rate by which these shocks disperse and the process reverts towards the mean, Qin (2011) and $\sigma^2$ is the instantaneous variance. The continuous component is given by a standard Brownian motion, $\mathrm{B}_t$ distributed as $d\mathrm{B}_t \sim \mathrm{N}(0,\ dt)$. The discontinuities of the price process are described by a Poisson counter $\mathrm{N}_t$, characterized by its intensity $\lambda$, and jump size $J_t$. The Brownian motion and the Poisson process are independent. The intensity of the Poisson process describes the mean number of arrivals of abnormal information per unit of time is expressed as $\Pr[\Delta\mathrm{N}_t = 1] = \lambda dt$ and $\Pr[\Delta\mathrm{N}_t = 0] = 1 - \lambda dt$ (Krienchene, 2006). When abnormal information arrives, crude oil price jumps from $S_{t-}$ to $S_t = \exp(J_t)S_{t-}$. The percentage change is measured by $\left(\exp(J_t) - 1\right)$. The jump size $J_t$, is independent of $\mathrm{B}_t$ and $\mathrm{N}_t$, and is assumed to be normally distributed $J_t \sim \mathrm{N}(\beta,\ \delta^2)$.

## METHODOLOGY

The paper uses the maximum likelihood estimation and the method of cumulants as main tool for estimating markets of commodities of crude oil price. Since the maximum likelihood of $\alpha$ in (14) is a simple transformation of the least squares estimator of the autoregressive coefficient in the first-order autoregressive (AR(1)) model with discrete data, the study intrinsically is related to the vast literature studying the finite-sample distribution of the AR(1) process. The model strategy involves identification, parameter estimation and diagnostic check, Box and Jenkins (1970). The maximum likelihood method and the method of cumulants adapted in estimating Jump-Diffusion model filters for crude oil price are:

(i) **The maximum likelihood method:** The maximum likelihood (ML) approach has appealing properties and is the most efficient under general conditions (Ibragimov and Hasminskii 1979). Let $x = \{x_1,\ x_2,\ \cdots,\ x_{\mathrm{T}}\}$ be an observed sample of log returns and f(x), the density, then the log-likelihood function can be expressed as:

$$L(\theta; x) = -\mathrm{T}\lambda - \frac{1}{2}\ln(2\pi) + \sum_{t=1}^{\mathrm{T}} \ln\left[\sum_{j=0}^{\infty} \frac{\lambda^n}{n!} \frac{1}{\sqrt{(\sigma^2 + n\delta^2)}} \exp\left(-\frac{(x-\mu-n\beta)^2}{2(\sigma^2+n\delta^2)}\right)\right] \quad (15)$$

Application of the maximum likelihood method for estimating the J-D model has met with difficulties arising mainly from the identification of the jump parameter and instability of parameter estimates (Krienchene, 2006). Nonetheless, Ball and Torous (1983) applied directly the ML method by truncating the number of jumps at $n$ = 15. They also applied the ML method by assuming a Bernoulli process for the jump component. While the ML estimates achieve the lower bound for Cramer-Rao efficiency criterion, difficulties with the likelihood function arising from computational tractability, un-boundedness over the parameter space, and instability of parameters, have led researchers to explore alternative estimation methods, based essentially on the method of moments.

(ii) **The method of cumulants:** Suppose that X is a real random variable whose real moment generating function is defined as $\mathrm{M}(u) = \mathrm{E}\left(e^{u\mathrm{X}}\right) = \int_{-\infty}^{\infty} e^{u\mathrm{X}} f(\mathrm{X}) d\mathrm{X}$, where $f(\mathrm{X})$ is the probability density of X. Just as the moment generating function M of X generates its moments, the logarithm of M generates a sequence of numbers called cumulants. The cumulants $K_n$ of the probability density of X are given by

$$\mathrm{M}(u) = \mathrm{E}\left(e^{u\mathrm{X}}\right) = 1 + \sum_{n=1}^{\infty} \frac{m_n u^n}{n!} = \exp\left(\sum_{n=1}^{\infty} \frac{K_n u^n}{n!}\right) \quad (16)$$

where $m_n = \mathrm{E}(\mathrm{X}^n)$ is the moment of order $n$ of X. The left-hand side of this equation is the moment-generating function, so $\frac{K_n}{n!}$ is the $n^{\text{th}}$ coefficient in the power series representation of the logarithm of the moment-generating function. The logarithm of the moment-generating function is therefore called the cumulant-generating function, written as $\log(\mathrm{M}(u)) = \sum_{n=0}^{\infty} \frac{K_n u^n}{n!}$. The cumulants are also equivalently defined in terms of the characteristic function, which is the Fourier transform of the probability density function: $\phi(u) = \mathrm{E}\left(e^{iu\mathrm{X}}\right) = \int_{-\infty}^{\infty} e^{iu\mathrm{X}} f(\mathrm{X}) d\mathrm{X}$. The cumulants $K_n$ are then defined as $\ln \phi(u) = \sum_{n=1}^{\infty} K_n \frac{(iu)^n}{n!}$. The method of cumulants attempts to recover a probability distribution from its sequence of cumulants. In some cases no solution exists; in some other cases a unique solution, or more than one solution, exists. The relationship between moments and cumulants is of paramount importance in the estimation of the unknown parameters of the density function. First, consider moments about 0, which can be written as $m_j = \mathrm{E}(\mathrm{X}^j), \quad j = 0,\ 1,\ 2,\ \cdots$. The cumulant/moment theorem says that if X is a random variable with *n* moments $m_1,\ m_2,\ \cdots,\ m_n$, then X has *n* cumulants $K_1,\ K_2,\ \cdots,\ K_n$, and the cumulants are related to the moments by the following recursion formula

$$K_n = m_n - \sum_{j=1}^{n-1} \binom{n-1}{j-1} K_n m_{n-j}. \tag{17}$$

This recursion formula is the Faa di Bruno's formula, equivalently written as

$$m_{r+1} = \sum_{j=0}^{r} \binom{r}{j} m_j k_{r+1} \quad \text{for } r = 0, 1, 2, \ldots n - 1. \tag{18}$$

Note that if the initial value $m_0 = 1$. By carrying the recursion formula, the relation between raw moments and cumulants can be stated as $m_1 = K_1$, respectively.

Press (1967) used the method of cumulants as described in Kendall and Stuart (1977) to estimate the J-D model. Define the characteristic function (CF) of $\mathrm{X}_t$ as

$$\phi_{\mathrm{X}}(u) = \mathrm{E}[\exp(iu\mathrm{X}_t)] = \int \exp(iu\mathrm{X}_t) f(\mathrm{X}_t) d\mathrm{X}_t \tag{19}$$

where $f(\mathrm{X}_t)$ is the probability density function of $\mathrm{X}_t$, *u* is the transform variable, and $\sqrt{-1} = i$. The characteristic function $\phi_{\mathrm{X}}(u)$ is related to the moment generating function $G_{\mathrm{X}}(u)$. $G_{\mathrm{X}}(u) = \mathrm{E}[\exp(u\mathrm{X}_t)] = \int \exp(u\mathrm{X}_t) dF(\mathrm{X}_t)$ by a change of the transform variable $u \to -iu$, namely $G_{\mathrm{X}}(iu) = \phi_{\mathrm{X}}(u)$ and $G_{\mathrm{X}}(u) = \phi_{\mathrm{X}}(-iu)$. The cumulants of $\mathrm{X}_t$, denoted by $K_n,\ n = 0,\ 1,\ 2,\ \cdots,$ are the coefficients in the power series expansion of the logarithm of the CF of $\mathrm{X}_t$, expressed as:

$$\ln \phi(u) = \sum_{n=1}^{\infty} K_n \frac{(iu)^n}{n!} = 1 + K_1 \frac{(iu)}{1!} + K_2 \frac{(iu)^2}{2!} + \cdots + K_n \frac{(iu)^n}{n!} + \cdots \qquad (20)$$

Noting that the characteristic function (CF) for the jump-diffusion process is given by:

$$\phi_{X_i}(u) = \exp\left[ -\frac{\sigma^2 u^2}{2} + i\mu u + \lambda \left( \exp\left( i\beta u - \frac{\delta^2 u^2}{2} \right) - 1 \right) \right]; \qquad (21)$$

It follows that the first four cumulants of the J-D process are $K_1 = \mu + \lambda\beta;\ K_2 = \sigma^2 + \lambda\delta^2 + \lambda\beta^2,$
$K_3 = \lambda\beta(3\delta^2 + \beta^2);\ K_4 = \lambda(3\delta^4 + 6\beta^2\delta^2 + \beta^4).$ Obviously, the cumulants enable to recover the parameters of the J-D process from sample moments. Press (1967), in order to avoid using higher order cumulants, imposed the restriction $\mu = 0$ and derived the following relations:

$$\hat{\beta}^4 - 2\frac{K_3}{K_1}\hat{\beta}^2 + \frac{3K_4}{2K_1}\hat{\beta} - \frac{K_3^2}{2K_1^2} = 0,\ \hat{\lambda} = \frac{K_1}{\hat{\beta}},\ \hat{\delta}^2 = \frac{K_3 - \hat{\beta}^2 K_1}{3K_1},\quad \hat{\sigma}^2 = K_2 - \frac{K_1}{\hat{\beta}}\left( \hat{\beta}^2 + \frac{K_3 - \hat{\beta}^2 K_1}{3K_1} \right). \qquad (22)$$

Press' estimates were often wrong-signed and not plausible. Beckers (1981) adopted the same method as Press, however, setting $\beta$, instead of $\mu$, to zero. Using sixth order cumulant, his cumulant equations yielded the following system:

$$\hat{\mu} = K_1,\quad \hat{\lambda} = \frac{25K_4^3}{3K_6^2},\quad \hat{\delta}^2 = \frac{K_6}{5K_4} \text{ and } \hat{\sigma}^2 = K_2 - \frac{5K_4^2}{3K_6}. \qquad (23)$$

Beckers' estimates improved those of Press, yet they were not free of anomalies. Ball and Torous (1983), using a Bernoulli, instead of a Poisson jump process and maintaining Beckers' restriction, i.e. $\beta = 0$, derived the following cumulant equations:

$$K_1 = \mu,\ K_2 = \sigma^2 + \lambda\delta^2,\ K_3 = 0,\quad K_4 = 3\delta^2\lambda(1-\lambda),\ K_5 = 0 \text{ and } K_6 = 15\delta^6\lambda(1-\lambda)(1-2\lambda). \qquad (24)$$

Again by equating with population cumulants, they obtained estimators $\hat{\mu},\ \hat{\lambda},\ \hat{\sigma}^2$ and $\hat{\delta}^2$ given by $\hat{\mu} = K_1$, $\hat{\lambda} = \frac{1 \pm \sqrt{3K^*/(3K^* + 100)}}{2},\ K^* = \left(\frac{K_6}{K_4}\right)^2$, $\hat{\sigma}^2 = K_2 - \hat{\lambda}\hat{\delta}^2$ and $\hat{\delta}^2 = \frac{K_6}{K_4(5(1-2\hat{\lambda}))}$. (25)

Das and Sundaram (1999) used the method of moments to estimate the J-D model. Denoting the log-price return by $x_t$ and assuming that the jump size $J_t$ is distributed as $J \sim \mathrm{N}(\beta,\ \delta^2)$, they showed that the first moments of the J-D process are given by the following equations which they used to estimate the model's parameters; however, for the Poisson parameter, $\lambda$, they imposed a given value.

$$Var(x) = \mathrm{E}\left[\left(x - \mathrm{E}(x)^2\right)\right] = \sigma^2 + \lambda\delta^2 \tag{26}$$

$$Skewness(x) = \frac{\mathrm{E}\left[\left(x - \mathrm{E}(x)\right)^3\right]}{\left[Var(x)\right]^{\frac{3}{2}}} = \frac{\lambda\left(\beta^3 + 3\beta\delta^2\right)}{\left(\sigma^2 + \lambda\delta^2 + \lambda\beta^2\right)^{\frac{3}{2}}} \tag{27}$$

$$Kurtosis(x) = \frac{\mathrm{E}\left[\left(x - \mathrm{E}(x)\right)^4\right]}{\left[Var(x)\right]^2} = 3 + \frac{\lambda\left(\beta^4 + 6\beta^2\delta^2 + 3\delta^4\right)}{\left(\sigma^2 + \lambda\delta^2 + \lambda\beta^2\right)^2} \tag{28}$$

# RESULTS AND DISCUSSION

Data on crude oil prices for January, 1986 to July, 2015 indicated that oil price volatility was not stationary (Figure 1). The dynamics were characterized by high volatility, high intensity jumps, and strong upward drift, indicating that oil markets were constantly out-of-equilibrium. The inverse leverage effect and the economic downturn experiencing in Nigeria from September, 2014 to the end of the sampled period are indications that the country's sole dependent on crude oil has consequences on non-provision of social amenities, public utilities, unemployment and by extension impoverishing the citizenry. This *"inverse leverage effect"* is also found in empirical studies for a large number of commodities such as oil, gas and soybeans, (Geman, 2005). While averaging 25 percent, volatility often surged to 40–45 percent, indicating that oil markets were experiencing big uncertainty regarding expected price developments and were highly sensitive to small shocks and news. Volatility pattern shows volatility clustering during rising pressure on oil prices and volatility decline during reduced pressure on oil prices. High volatility increases speculative demand for futures contracts, which in turn leads to higher volatility and volatility clustering. The plot of log-transform of the data on monthly oil prices covering January, 1986 to July, 2015 gives the oil price return defined as: $x_t = \Delta \log S_t = \log S_t - \log S_{t-1}$ in Figure 2. The fitting of the GARCH model showed high price volatility, periods of volatility clustering, followed by some reversion to a mean volatility estimated at 33 percent. GARCH volatility was rising during periods of large price shocks, increasing speculation and leading to volatility clustering; it was, however, receding during periods of price retreat. It corroborated the observed implied volatility, namely oil markets were constantly experiencing large uncertainties and were impacted by frequent shocks.

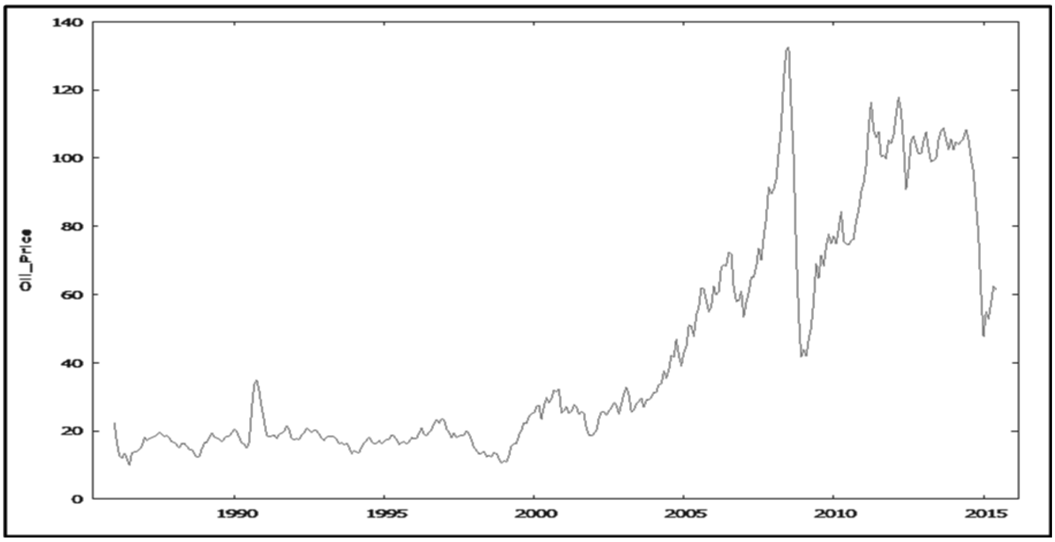


Fig. 1: Time Plot of Monthly Crude Oil Price (January, 1986-July, 2015) data

Obviously, the graph of oil price is not stationary. Sharp oil price movements, dramatic uncertainty for the global economy and trends in changing oil prices has an impact on world politics, economy, military and all sectors of society, especially in Nigeria and other developing countries and therefore threaten future oil demand growth. So increasingly important is the interests of government, companies and investors on this crude oil. The plot details many striking facts regarding oil markets at the outset. Foremost, world oil demand pressure kept increasing within the sampled period, causing oil prices to rise by more than three-fold, January, 2002 to October, 2009.  Second, the noted ascent in oil prices was not monotonic or smooth; oil prices rose, often to a new record, retreated temporarily, then resumed their move to higher record; their movements were dominated by high intensity jumps, indicating that oil markets were constantly out-of-equilibrium. Third, oil price volatilities were excessively high. Measured by the implied volatility, volatility was in the range of 21 percent, implying that oil markets were facing big uncertainties regarding future price developments and were sensitive to small shocks and to news. More specifically, markets held higher probabilities for further price increases than price decreases. Recently, the world economy is experiencing recession and this lead to a drastic fall in crude oil prices as opposed to what happened from 2000 to 2009.

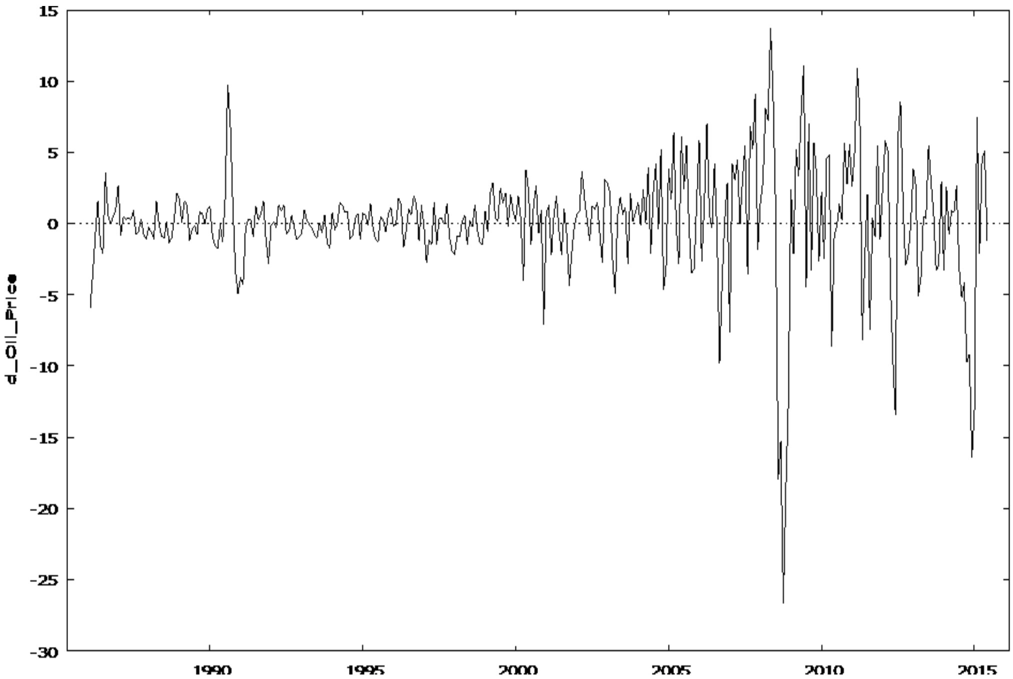


Fig. 2: Volatility Clustering of log Monthly Crude Oil Price (January, 1986-July, 2015) data

We observe from Figure 2 that there exist a prolonged low volatility between1960 to 1991 and then there is a sudden shock, spike which is immediately followed by a prolonged low volatility. In other words, periods of high volatility is followed by periods of low volatility and periods of low volatility is followed by periods of high volatility (Mandelbrot 1963). Figure 2 also depicts that the distribution was left-skewed (with estimate of -0.35), implying that downward jumps of smaller size were more frequent than upward jumps of larger size; as the mean was positive and high, smaller jumps were outweighed by larger jumps. The distribution had also fat tails, meaning that large jumps tended to occur more frequently than in the normal case (kurtosis equals 4.17 greater than 3). When the residual behaves like this, it suggests that the error term or residual is conditionally heteroskedastic and it can be represented by autoregressive conditional heteroskedasticity (ARCH) and generalized autoregressive conditional heteroskedasticity (GARCH) family models.

Let $S_t$ be the oil futures price in US$/bl. An augmented Dickey-Fuller test indicated that $S_t$ possessed a unit root; it was pulled by an upward trend, showing no sign for mean reversion. Changes in $S_t$, defined as $\Delta S_t = S_t - S_{t-1}$ , were, however, stationary. Based on the unit-root test, the dynamics of the oil process were represented by a simple auto-regression of order two (AR2)

$$\begin{array}{l} S_t = 0.89S_{t-1} + 0.045S_{t-2} + 0.12 \\ (t = 29.4) \quad (t = 3.6) \quad (t = 0.78) \end{array} \tag{29}$$

which yielded good fit $(R^2 = 0.96)$, highly significant coefficients as evident from high t statistics and the DW = 2.03.

Based on a sample of monthly crude oil prices, alternative methods of Krienchene (2006) were also used for estimating the J-D model. First, assuming a Bernoulli jump process, the ML was applied unrestrictedly, and with restriction on the probability λ of a jump occurring on a trading day given by $\lambda = 0.20$. Second, the method of cumulants was applied consecutively with restrictions $\lambda = 0.20,\ \mu = 0$ (Press, 1967) and $\beta = 0$ (Beckers, 1981), respectively. The results were approximately in agreement with Krienchene (2006) estimates as presented in the Table 1 below.

**Table 1: Jump-Diffusion Model: Parameter Estimates**

| Methods | Drift $\mu$ | Variance $\sigma^2$ | Intensity $\lambda$ | Mean $\beta$ | Variance $\delta^2$ |
|---|---|---|---|---|---|
| Bernoulli process | | | | | |
| Maximum likelihood | 0.21<br>(t = 3.22) | 6.53<br>(t = 21.43) | 0.69<br>(t = 1.76) | -2.17<br>(t = -3.19) | 6.54<br>(t = 18.75) |
| Maximum likelihood 1/ | 0.29<br>(t =3.18) | 5.74<br>(t =19.24) | 0.20 | -0.57<br>(t = -2.77) | 11.89<br>(t = 9.31) |
| Cumulants 1/ | 0.33 | 1.91 | 0.20 | -0.73 | 3.47 |
| Press (1967) 2/ | 0 | 6.97 | 0.13 | 1.15 | -12.68 |
| Beckers (1981) 3/ | 0.10 | 3.46 | 0.27 | 0 | 9.21 |

1/ Restriction on $\lambda = 0.20$, computed from the data sample as the frequency of a jump in the crude oil price
exceeding $\pm 3$ percent. 2/ Restriction on $\mu = 0$. 3/ Restriction on $\beta = 0$.

In Table 1, assuming a Bernoulli jump process, the ML estimates were highly significant and stable. The drift of the diffusion component, estimated at $\hat{\mu} = 0.21$, was very high and significant, showing that oil prices were constantly under pressure to move upward. The variances of the diffusion and jump components were high and significant, $\hat{\sigma}^2 = 6.53$ and $\hat{\delta}^2 = 6.54$, respectively. The variance of the jump component became more important than that of the diffusion component when the jump intensity was restricted to $\lambda = 0.20$. The probability of a jump in the unrestricted case, computed at $\hat{\lambda} = 0.69$, was high and borderline significant. The mean of the jump component, estimated at $\hat{\beta} = -2.17$, was negative and consistent with the negative skewness (-0.35) observed in the data. Oil prices tended to make large moves upward, then started to retreat through a sequence of smaller and frequent negative jumps, until they were shocked again, making new jumps forward. Yet, the significance of the drift of the diffusion process was such that the smaller negative jumps could not outweigh the strong momentum that kept pushing oil prices upward.

The method of cumulants was applied under alternative restrictions. The restriction $\lambda = 0.20$ yielded results that were similar to the ML under the same restriction. The drift of the diffusion component, estimated at $\hat{\mu} = 0.33$, was very high, showing that oil prices were constantly under pressure to move upward. The variances of the diffusion and jump components, were estimated at $\hat{\sigma}^2 = 1.91$ and $\hat{\delta}^2 = 3.47$, respectively, indicating that the jump component tended to dominate the dynamics of the oil price process. The mean of the jump component, estimated at $\hat{\beta} = -0.73$, was negative and consistent with the negative skewness in oil price returns. Application of the Press (1967) method, with the restriction $\mu = 0$, yielded implausible results for the variance of the jump component, namely $\hat{\delta} = -12.68\,2$. Such an anomaly was not unexpected in the case of Press' method, indicating that the restriction $\mu = 0$, could not be borne by the data, and was in sharp contrast with the strong upward trend in oil prices. In contrast, Beckers' method, with the restriction $\beta = 0$, yielded results which were highly plausible. The drift component of the diffusion, estimated at $\hat{\mu} = 0.10$, was smaller than, say, in the ML case, since $\beta = 0$ implied less influence for the drift of the diffusion, compared to the case when $\beta$ was negative, to maintain an upward trend in oil prices; it was, however, close to the drift of the AR(2) model in equation (29). The variances of the diffusion and jump components were high, $\hat{\sigma}^2 = 3.46$ and $\hat{\delta}^2 = 9.21$, respectively. The variance of the jump component, however, dominated that of the diffusion component. Noticeably, the jump intensity, estimated at $\hat{\lambda} = 0.27$, was quite close to the frequency of jumps in oil prices exceeding ±3 percent, computed from the data set.

The results obtained established that when modelled crude oil price as a jump-diffusion (J-D) process, oil price dynamics were dominated by the discontinuous

Poisson jump component compared to the continuous Gaussian diffusion component, showing that oil markets were constantly out-of-equilibrium during the sample period and were sensitive to demand and supply shocks and to news. While the variance of the diffusion component was high and significant, it was surpassed by a still higher and significant variance of the jump component. Both variances, together, illustrated the high volatility of the oil markets.

## CONCLUSION

The empirical analysis indicates that the autoregressive model of order two (AR(2)), generalized autoregressive conditional heteroskedasticity (GARCH (1, 1)) model as well as the Jump-Diffusion model are best for estimating the crude oil price data series within the sampled period. The results also show that the Jump-Diffusion model out-perform the other two models for estimating the drift and jump despite the crude oil fluctuations. The method of maximum likelihood and cumulants were utilized. The results show that the drift of the diffusion component $(\mu)$ was, however, very high and significant, indicating that oil prices were strongly influenced by an upward trend. The mean of the jump component $(\beta)$ was negative; more specifically, sharp upward jumps in oil prices had a temporary restraining effect on oil demand and were followed by a short-lived sequence of price retreats. The mean of the jump component was, however, outweighed by the drift of the diffusion component which kept prices on a rising trajectory. The study, therefore, provides independent and impartial crude oil price information to promote sound policymaking, efficient markets and understanding of crude oil price and its interaction with the economy and the environment.

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